

# Compressed entanglement testing

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We introduce a sequence of numerical tests that can determine the entanglement or separability of a state even when there is not enough information to completely determine its density matrix. Given partial information about the state in the form of linear constraints on the density matrix, the sequence of tests can prove that either all states satisfying the constraints are entangled, or that there is at least one separable state that satisfies them. The algorithm works even if the values of the constraints are only known to fall in a certain range. If the states are entangled, an entanglement witness is constructed and lower bounds on entanglement measures and related quantities are provided; if a separable state satisfies the constraints, a separable decomposition is provided to certify this fact.

Entanglement is one of the central features in quantum information processing (QIP). It has been identified as a key ingredient in many useful QIP tasks such as teleportation, quantum key distribution, superdense coding and quantum computation [1]. It is also remarkable that even when a complete description of a quantum state is given (in the form of a density matrix), it can be extremely difficult to computationally decide whether such state is entangled or not. This is due to the fact that this problem (usually called “the separability problem”) is known to be NP-Hard [2]. The problem is even more difficult when we consider experimental tests of entanglement, since measurements may not provide a full description of the state, and when they do (such as in quantum state tomography [3]) the reconstructed density matrix may be unphysical (i.e., not positive semidefinite (PSD)).

A key problem with important practical applications is to determine the entanglement characteristics of a state when only a limited amount of information is available. If this information comes from measuring a set of observables, it takes the form of a set of linear constraints on the elements of the density matrix. This set of constraints may also come from other more theoretical considerations. In this letter we will introduce a sequence of numerical tests that can decide whether all states that satisfy a set of linear constraints are entangled, or if there is at least one separable state that satisfy those same constraints. If the states are shown to be entangled, the algorithm constructs an entanglement witness that certifies this fact for all such states and provides lower bounds on certain entanglement measures and related quantities. If a separable state satisfying the constraints exists, the algorithm finds it and provides a separable decomposition as a proof.

Consider a situation in which we are given partial information of the state of a quantum system in the form of

$L$  linear constraints on the elements of its density matrix

$$\text{Tr}[\rho M_l] = m_l, \quad l = 1, \dots, L \quad (1)$$

where the operators  $M_l$  are arbitrary. Our goal is to determine if all the states satisfying these constraints are entangled, or if there is a separable state that satisfies them. The constraints in (1) are nothing but a linear system of equations for the elements of the density matrix  $\rho$ . If this system is incompatible it means that these constraints do not describe a physical state. If the system is invertible, then the Hermitian matrix  $\rho$  can be completely determined from the equations, and once we have an explicit expression we can check if it corresponds to a state (i.e., it is PSD and normalized), and then apply any of a battery of separability criteria. In particular, if we apply the combined PPT symmetric extension (PPTSE) criterion introduced in [4] and its dual introduced in [5], we know that we can conclusively decide the entanglement properties of the state (although it may be computationally very expensive for some states). But the situation that is the most interesting (and typically more common) corresponds to the case in which the linear system (1) is underdetermined, and we do not have enough information to uniquely define the state. This situation corresponds naturally to being able to measure only the expectation value of a limited number of observables (the operators  $M_l$  in (1) are then Hermitian matrices). We will show that in this case, the linear system defines an affine subspace in the state of Hermitian matrices, and the PPTSE criterion and its dual can be applied to either prove that all states in that affine subspace are entangled, or to show that a separable state exists that satisfies (1).

Let us start with the linear system (1). The most general solution  $\rho$  of this system can be written as

$$\rho = \rho^{part} + \sum_{a=1}^{D_K} y_a \mu^{(a)}, \quad (2)$$

where  $\rho^{part}$  is a particular solution of (1) (i.e.,  $\text{Tr}[\rho^{part} M_l] = m_l$ ,  $l = 1, \dots, L$ ), the matrices  $\{\mu^{(a)}\}$

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form a basis of the subspace of solutions of the homogeneous system (i.e.,  $\text{Tr}[\mu^{(a)} M_l] = 0$ ,  $l = 1, \dots, L$ ),  $D_K$  is the dimension of this subspace, and  $y_a$  are real variables. Note that  $\rho^{part}$  is just a Hermitian matrix and not necessarily a state (i.e., it need not be PSD or normalized). The question then reduces to whether there are values of the real variables  $y_a$  such that the resulting Hermitian matrix is a normalized, separable state. If there are not, then all the states of the form (2), that is all normalized PSD Hermitian matrices satisfying (1) must be entangled.

In order to test the separability of (2) we will use the PPTSE criterion [6]. We define a PPT symmetric extension of  $\rho$  to  $k$  copies of subsystem  $A$  as a state  $\tilde{\rho}$  in  $\mathcal{H}_A^{\otimes k} \otimes \mathcal{H}_B$  such that, (i)  $\rho = \text{Tr}_{A^{k-1}}[\tilde{\rho}]$ , (ii)  $\tilde{\rho}$  is symmetric under exchanges of copies of subsystem  $A$ , and (iii)  $\tilde{\rho}$  has positive partial transposes for any bipartite arrangement of the subsystems  $A$  and  $B$ . Since any separable state in  $\mathcal{H}_A \otimes \mathcal{H}_B$  can be written as  $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$ , it has such an extension given by  $\tilde{\rho} = \sum p_i |\psi_i\rangle\langle\psi_i|^{\otimes k} \otimes |\phi_i\rangle\langle\phi_i|$ . For each value of  $k$ , the non-existence of a PPTSE provides a sufficient (but not necessary) condition for entanglement. In the limit  $k \rightarrow \infty$  the condition becomes necessary. The practical value of this approach is that searching for such extensions or proving their impossibility can be cast as a semidefinite program.

A semidefinite program (SDP) is a type of convex optimization problem that has a broad range of applications and has been widely applied in quantum information. An SDP has both a primal and a dual form. A typical SDP in its primal form reads

$$\begin{aligned} & \text{minimize} && c^T \mathbf{x} \\ & \text{subject to} && F_0 + \sum_i x_i F_i \succeq 0, \end{aligned} \quad (3)$$

where  $c$  is a given vector,  $\mathbf{x} = (x_1, \dots, x_n)$ , and  $F_0$  and  $F_i$  are some fixed Hermitian matrices. The inequality in the second line means that the affine combination of the  $F$  matrices must be positive semidefinite. The minimization is performed over the vector  $\mathbf{x}$ , whose components are the variables of the problem. The dual of this SDP takes the form

$$\begin{aligned} & \text{maximize} && -\text{Tr}[F_0 Z] \\ & \text{subject to} && Z \succeq 0, \\ & && \text{Tr}[F_i Z] = c_i, \end{aligned} \quad (4)$$

where the dual variables are the components of the matrix  $Z$ .

To test the entanglement of a state of the form (2), we can apply the PPTSE criterion for any value of  $k$ . We will present in detail how this works for  $k = 2$  (the general case is straightforward). So we need to check if, for some values of the variables  $y_a$ , the resulting matrix is PSD, normalized and has a PPTSE. Let  $\{\sigma_i^A\}_{i=1}^{d_A^2}, \{\sigma_j^B\}_{j=1}^{d_B^2}$  be bases for the spaces of Hermitian matrices that operate on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , of dimensions  $d_A$  and  $d_B$  respectively, such that they satisfy  $\text{Tr}[\sigma_i^X \sigma_j^X] = \alpha_X \delta_{ij}$  and

$\text{Tr}[\sigma_i^X] = \delta_{i1}$  (where  $X$  stands for  $A$  or  $B$ ), and  $\alpha_X$  is some constant. Then we can expand  $\rho$  in the basis  $\{\sigma_i^A \otimes \sigma_j^B\}$ , and write  $\rho = \sum_{ij} \rho_{ij} \sigma_i^A \otimes \sigma_j^B$ , with  $\rho_{ij} = \alpha_A^{-1} \alpha_B^{-1} \text{Tr}[\rho \sigma_i^A \otimes \sigma_j^B]$ . In the same way, we can expand the extension  $\tilde{\rho}$  in  $\mathcal{H}_A^{\otimes 2} \otimes \mathcal{H}_B$  as

$$\begin{aligned} \tilde{\rho} = & \sum_{\substack{ikj \\ i < k}} \tilde{\rho}_{ikj} \{\sigma_i^A \otimes \sigma_k^A \otimes \sigma_j^B + \sigma_k^A \otimes \sigma_i^A \otimes \sigma_j^B\} + \\ & + \sum_{kj} \tilde{\rho}_{kkj} \sigma_k^A \otimes \sigma_k^A \otimes \sigma_j^B, \end{aligned} \quad (5)$$

where we made explicit use of the swapping symmetry between the two copies of  $A$ . To satisfy the condition that  $\tilde{\rho}$  is an extension of  $\rho$ , we need to impose  $\text{Tr}_A[\tilde{\rho}] = \rho$ . This implies  $\tilde{\rho}_{i1j} = \rho_{ij}$ . From (2) we have that  $\rho_{ij} = \rho_{ij}^{part} + \sum_{a=1}^{D_K} y_a \mu_{ij}^{(a)}$ , which fixes some of the components of the extension (5). We then have

$$\begin{aligned} \tilde{\rho} = & \sum_{\substack{ij \\ i > 1}} \rho_{ij}^{part} \{\sigma_i^A \otimes \sigma_1^A \otimes \sigma_j^B + \sigma_1^A \otimes \sigma_i^A \otimes \sigma_j^B\} + \\ & + \sum_j \rho_{1j}^{part} \sigma_1^A \otimes \sigma_1^A \otimes \sigma_j^B + \\ & + \sum_{a=1}^{D_K} y_a \left( \sum_{\substack{ij \\ i > 1}} \mu_{ij}^{(a)} \{\sigma_i^A \otimes \sigma_1^A \otimes \sigma_j^B + \sigma_1^A \otimes \sigma_i^A \otimes \sigma_j^B\} + \right. \\ & \left. + \sum_j \mu_{1j}^{(a)} \sigma_1^A \otimes \sigma_1^A \otimes \sigma_j^B \right) + \\ & + \sum_{\substack{ijk \\ i > k \geq 2}} \tilde{\rho}_{ikj} \{\sigma_i^A \otimes \sigma_k^A \otimes \sigma_j^B + \sigma_k^A \otimes \sigma_i^A \otimes \sigma_j^B\} + \\ & + \sum_{\substack{jk \\ k \geq 2}} \tilde{\rho}_{kkj} \sigma_k^A \otimes \sigma_k^A \otimes \sigma_j^B, \end{aligned} \quad (6)$$

If we define a vector of variables  $\mathbf{x} = (\mathbf{y}, \tilde{\rho}_{ikj})$  (with  $2 \leq k \leq i \leq d_A^2, 1 \leq j \leq d_B^2$ ), we can see that the most general form of the extension (6) has the form  $G_0 + \sum_i x_i G_i$ , where the expressions for  $G_0$  and  $G_i$  can be easily extracted from it.

The first condition we need to impose on this extension is that it represents a state, i.e., that it is PSD and normalized. The normalization condition can always be assumed to be contained in the set of linear equations (1), by adding another constraint with  $M = \mathbf{1}$  and expectation value equal to 1. Requiring that the extension is PSD means imposing the linear matrix inequality (LMI)  $G_0 + \sum_i x_i G_i \succeq 0$ . And finally, imposing the positivity of the partial transposes requires two more LMIs, namely  $G_0^{T_A} + \sum_i x_i G_i^{T_A} \succeq 0$  and  $G_0^{T_B} + \sum_i x_i G_i^{T_B} \succeq 0$  (due to the swapping symmetry, these are the only two independent partial transposes). We can combine these three LMIs into a single one by defining matrices  $F_0 = G_0 \oplus G_0^{T_A} \oplus G_0^{T_B}$  and

$F_i = G_i \oplus G_i^{T_A} \oplus G_i^{T_B}$  (a block diagonal matrix is PSD if and only if all of its blocks are PSD). So searching for a PPTSE of a state of the form (2) corresponds to a SDP of the form (3) with  $c = (0, \dots, 0)$ . If there are values of  $y_a$  such that (2) is separable, then there must exist values of  $\tilde{\rho}_{ikj}$ , ( $2 \leq k \leq i \leq d_A^2$ ,  $1 \leq j \leq d_B^2$ ) such that the SDP is feasible (because separable states always have PPTSE). But if the SDP is infeasible, it means that there is no set of values  $y_a$  for which the resulting state  $\rho$  has a PPTSE, and hence all states of the form (2) must be entangled.

We can take this approach a little bit further and consider the case in which the expectation values of the operators  $M_l$  (the RHS of (1)) are known only approximately. Assume that instead of (1) we have  $\text{Tr}[\rho M_l] \in [m_l^{\min}, m_l^{\max}]$ ,  $l = 1, \dots, L$ . Now consider the set of matrices  $\{\tau_p : \text{Tr}[\tau_p M_l] = \delta_{pl}\}$ . We can use these matrices to write any particular solution of the linear system  $\text{Tr}[\rho M_l] = z_l$  as  $\rho^{\text{part}} = \sum_{l=1}^L z_l \tau_l$ . Combining this with (2) we have

$$\rho = \sum_{l=1}^L z_l \tau_l + \sum_{a=1}^{D_K} y_a \mu^{(a)}, \quad (7)$$

as the most general solution of  $\text{Tr}[\rho M_l] \in [m_l^{\min}, m_l^{\max}]$ ,  $l = 1, \dots, L$ , provided that  $z_l \in [m_l^{\min}, m_l^{\max}]$ . If we apply the PPTSE criterion to (7) as before, we will obtain a linear combination of matrices representing the PPT symmetric extension of a state satisfying this set of equations. We just need to once again construct the required LMIs to impose the positive semidefiniteness of the extension and its partial transposes, and solve the resulting SDP satisfying the constraints  $z_l \in [m_l^{\min}, m_l^{\max}]$ . These constraints can be imposed by another LMI, namely  $\text{diag}(z_1 - m_1^{\min}, m_1^{\max} - z_1, \dots, z_L - m_L^{\min}, m_L^{\max} - z_L) \succeq 0$ , showing that constraining the range of the variables does not change the SDP structure.

Another useful feature of the PPTSE criterion is that, if the primal SDP is infeasible (i.e., there is no separable state satisfying the constraints), the dual SDP provides a certificate of this fact in the form of an entanglement witness [6]. Let us recall that an entanglement witness is a Hermitian operator whose expectation value is positive on all separable states but negative on some entangled states. It provides a proof of the entanglement of a given state. Entanglement witnesses are a particular case of the separating hyperplane theorem (or Hahn-Banach theorem) of convex geometry: if two closed convex sets are disjoint and one of them is compact, there is a hyperplane that separates them (i.e., the sets are in opposite sides of the hyperplane). In the context of checking separability of linearly constrained states, the convex sets in question are the set of separable states and the affine subspace spanned by all the solutions of the linear system (1). If these two sets are disjoint, it means that no separable state satisfies (1); on the other hand, the separating hyperplane theorem assures us that there is an entanglement witness that can certify the entanglement

of every state of the form (2).

A slight modification of the SDP allows us to compute bounds on certain useful entanglement quantities. Let us add an extra variable  $t$  and an extra term to the LMI:  $F_0 + \sum_i x_i F_i + t \mathbf{1} \succeq 0$ . This problem is equivalent to the original if we minimize  $t$ : if the original is feasible,  $t_{\text{opt}} \leq 0$ , and  $t_{\text{opt}} > 0$  if it is infeasible. But then  $d_A^2 d_B t_{\text{opt}}$  is a lower bound on the minimum amount of the maximally mixed state we need to add to a state satisfying (1) to make it separable (in  $2 \otimes 2$  and  $2 \otimes 3$  this bound is tight). This is known as the random robustness of entanglement [7],  $R_r(\rho)$ , and quantifies how robust the entanglement is against white noise. It also provides a lower bound on a geometric measure of quantum discord [8].

The entanglement witness constructed from the dual SDP can also be used to quantify the entanglement of the states satisfying (1). Any entanglement measure that can be expressed as  $E(\rho) = \max\{0, \min_{W \in \mathcal{M}} \text{Tr}[W\rho]\}$  with  $\mathcal{M}$  a subset of entanglement witnesses, is referred to as *witnessed entanglement* [9]. The set  $\mathcal{M}$  determines which particular measure this expression represents. Several well-known measures are of this form, such as the best separable approximation  $BSA(\rho)$ , the negativity  $\mathcal{N}(\rho)$ , and the concurrence  $\mathcal{C}(\rho)$ . Clearly, any  $W \in \mathcal{M}$  that satisfies  $\text{Tr}[W\rho] < 0$  provides a lower bound to  $E(\rho)$ . In particular, the quantities  $E_{n,m}(\rho) = \max\{0, \min_{W \in \mathcal{M}_{n,m}} \text{Tr}[W\rho]\}$  ( $n, m \geq 0$ ) with  $\mathcal{M}_{n,m} = \{W : -n\mathbf{1} \preceq W \preceq m\mathbf{1}\}$  a subset of entanglement witnesses, are entanglement monotones, and satisfy  $E_{n,m}(\rho) \rightarrow nBSA(\rho)$  when  $m \rightarrow \infty$ , where  $BSA(\rho)$  is the best separable approximation to  $\rho$  [9]. Since  $E_{n,m}(\rho)$  is obviously monotonically increasing with  $m$  (for fixed  $n$ ) and any entanglement witness  $Z$  must be in some  $\mathcal{M}_{n,m}$ ,  $\text{Tr}[Z\rho]$  provides a lower bound on  $BSA(\rho)$ , which is an entanglement measure. This analysis is just an illustration of the connection between the PPTSE criterion and entanglement measures, and does not pretend to give the best bounds possible.

Each test in the PPTSE hierarchy provides a sufficient but not necessary condition for entanglement. The hierarchy is complete in the limit: any entangled state is guaranteed to be detected by one of the tests. But a separable state will pass all tests, leading to a non-terminating algorithm. Fortunately, a dual approach was developed by Navascués et al., [5], that applies a sequence of tests that can certify separability in a finite number of steps (although that number can be very high for some states). Geometrically, the PPTSE hierarchy of tests works by monotonically approximating the cone of separable states from the outside with a sequence of cones associated with states having PPT symmetric extensions to a certain number of copies of one of the subsystems. The dual approach in [5] constructs a similar approximation to the cone of separable states, but from the inside: it provides sufficient (but not necessary) conditions for separability. By interleaving the two sequences of tests we can, in a finite number of steps, determine if a state is

entangled (and give an entanglement witness as a proof), or separable (and provide an explicit separable decomposition). We will now briefly describe the test in [5] and show how it can be used to determine if there is a state of the form (2) that is separable.

Let  $\mathcal{S}_p^N$  be the set of states in  $\mathcal{H}_A \otimes \mathcal{H}_B$  that have a PPT symmetric extension to  $N$  copies of  $A$ . In [10] it was shown that a small perturbation in  $\mathcal{H}_B$  makes these states separable. More precisely we have that  $\tilde{\mathcal{S}}_p^N \equiv \{(1 - \epsilon_N)\omega_{AB} + \epsilon_N \omega_A \otimes \frac{1}{d_B} : \omega_{AB} \in \mathcal{S}_p^N\}$  satisfies  $\tilde{\mathcal{S}}_p^N \subset \mathcal{S}$ , for all  $N$ , where  $\mathcal{S}$  is the set of separable states in  $\mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\omega_A = \text{Tr}_B[\omega_{AB}]$ , and  $\epsilon_N \equiv d_B/(2(d_B - 1))\min\{1 - x : P_{\lfloor N/2 \rfloor + 1}^{(d_B - 2, N \bmod 2)}(x) = 0\}$ , with  $P_n^{(\alpha, \beta)}(x)$  the Jacobi polynomials. Since  $\tilde{\mathcal{S}}_p^N \rightarrow \mathcal{S}_p^N$  for  $N$  going to infinity, and  $\mathcal{S} \subset \mathcal{S}_p^N$  for all  $N$ , we have that  $\tilde{\mathcal{S}}_p^N \rightarrow \mathcal{S}$  ( $N \rightarrow \infty$ ). This result can be easily transformed into a SDP that tests if a given state is separable [5]. The same approach applies to states defined by (2), and so we can construct a sequence of tests that is guaranteed to find a separable state that satisfies (1) if one exists. Moreover, an explicit separable decomposition will be provided in that case (the actual form can be found in [10]).

Let us use a simple example to illustrate the power of this approach. Consider a system that produces two photons and we want to determine if they are entangled in the polarization basis. One possible approach is to do quantum state tomography. This can be accomplished by measuring the 16 observables given by [11]  $\hat{\mu}_i \otimes \hat{\mu}_j$  ( $i, j = 0, 1, 2, 3$ ) with  $\hat{\mu}_0 = |H\rangle\langle H| + |V\rangle\langle V|$ ,  $\hat{\mu}_1 = |H\rangle\langle H|$ ,  $\hat{\mu}_2 = |D\rangle\langle D|$ ,  $\hat{\mu}_3 = |R\rangle\langle R|$ , with  $|D\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$  and  $|R\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}$ . Note that these operators are all positive on pure product states and so they are good candidates to be entanglement witnesses. Applying our test we find an example in which only four of these observables are enough to prove entanglement: if  $\text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_1)\rho] \in [0.48, 0.5]$ ,  $\text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_2)\rho] \in [0.24, 0.25]$ ,  $\text{Tr}[(\hat{\mu}_2 \otimes \hat{\mu}_2)\rho] \in [0.48, 0.5]$ ,  $\text{Tr}[(\hat{\mu}_3 \otimes \hat{\mu}_3)\rho] \in [0, 0.02]$ , (note that all expectation values are non negative), there is an entanglement witness  $Z = 0.1343|HH\rangle\langle HH| + 0.3977|HV\rangle\langle HV| + 0.234(|VH\rangle\langle VH| + |VV\rangle\langle VV|) + \{(0.0658 + i0.1583)(|HH\rangle\langle VH| + |HV\rangle\langle VV| + |VH\rangle\langle VV|) +$

$h.c.\} + \{-0.2242|HH\rangle\langle VV| + 0.0925|HV\rangle\langle VH| + h.c.\}$ , such that  $\text{Tr}[Z\rho] < -0.0168$  for all states satisfying the linear constraints above. Moreover,  $Z \in \mathcal{M}_{1,1}$ , so this result provides a lower bound on the best separable approximation, i.e.,  $BSA(\rho) \geq 1.68 \times 10^{-2}$ . The primal SDP also computes a lower bound on the random robustness,  $R_r(\rho) \geq 8 \times 0.0168 = 0.1344$ . Additional information about the state can improve these bounds. For example, if we add  $\text{Tr}[(\hat{\mu}_1 \otimes \hat{\mu}_3)\rho] \in [0.24, 0.25]$  to the constraints we now obtain a new entanglement witness  $Z'$  such that  $\text{Tr}[Z'\rho] < -0.021$ , which translates to  $BSA(\rho) \geq 2.1 \times 10^{-2}$ , and  $R_r(\rho) \geq 0.168$  (the MATLAB code used is available online from the author [12]).

An interesting point to consider is when does this combined approach fail, i.e., it does not produce a definite answer after a reasonable number of steps. If we look at the PPTSE part of the method (proving entanglement), we can see that if the affine space defined by (2) intersects sets  $\mathcal{S}_p^N$  with  $N$  large, even if the states are actually entangled it will take at least  $N$  steps to prove this fact. On the other hand, if we consider the dual approach and (2) intersects the set of separable states but does not intersect  $\tilde{\mathcal{S}}_p^N$  for  $N$  small, the procedure will take a long time to provide the required separable decomposition. Since  $\mathcal{S}_p^N$  and  $\tilde{\mathcal{S}}_p^N$  approach  $\mathcal{S}$  from the outside and the inside respectively, the most challenging situation for this procedure occurs when (2) is “almost” tangent to the set of separable states, either intersecting it or not. Failure of the approach to provide an answer in a reasonable number of steps suggests that stronger constraints on the state may be needed.

In summary, the algorithm described in this letter allows us to determine the entanglement character of states even when only partial information about them is available. It also provides bounds on measures of entanglement and other related quantities. This tool can thus be very useful to experimentally determine if a state is entangled when the number of observables that can be measured is limited.

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- [1] M. N. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000).
  - [2] L. Gurvits, in *STOC '03: Proceedings of the thirty-fifth annual ACM symposium on Theory of computing* (ACM Press, New York, NY, USA, 2003), pp. 10–19.
  - [3] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
  - [4] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002).
  - [5] M. Navascués, M. Owari, and M. B. Plenio, Phys. Rev. Lett. **103**, 160404 (2009).
  - [6] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A **69**, 022308 (2004).
  - [7] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
  - [8] T. Debarba, T. O. Maciel, and R. O. Vianna, arxiv:1207.1298v1 (unpublished).
  - [9] F. G. S. L. Brandão, Phys. Rev. A **72**, 022310 (2005).
  - [10] M. Navascués, M. Owari, and M. B. Plenio, Phys. Rev. A **80**, 052306 (2009).
  - [11] D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A **64**, 052312 (2001).
  - [12] [http://drop.isi.edu/people/fspedali/entanglement\\_code](http://drop.isi.edu/people/fspedali/entanglement_code).